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An Alternative Linear Function for Standard Form of Cobb Douglas Production Function

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Abstract:

The objective of this paper is to find out an alternative linear function for the standard form of Cobb Douglas production function that can be estimated by using ordinary least squares (OLS) or other method of estimation. We also illustrate our finding with numerical example.

We concluded that the new transformation model is an alternate for the tradition log transformation model , and the new transformation model doesn't involve a constant term.

Keywords: Cobb Douglas, Production function, Linear function, Ordinary Least Squares, dummy variables .

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الملخص بالعربية

تهدف هذه الورقة العلمية لايجاد طريقة بديلة لطريقة اللوغريثم الطبيعي التقليدية المعروفة لتحويل دالة كوب دوقلاس الغير خطية الى دالة خطية من اجل تقديرها باستخدام طريقة المربعات الصغرى الاعتيادية أو أي طريقة تقدير أخرى، كما تم تدعيم الورقة بمثال تطبيقي من اجل المقارنة بين طريقة التحويل المقترحة وطريقة التحويل باستخدام اللوغريتم الطبيعي التقليدية.

وقد تم التوصل الى ان النموذج المقترح هو بديل لنموذج اللوغريتم التقليدي، وان النموذج المقترح لا يحتوي على الحد الثابت.

1. Introduction:

The simplest production function used frequently in economics is a Cobb-Douglas , In its standard form for production of a single good with two factors, the function is [1,2,3]:

$$Q_{it} = A_i L^{\alpha}_{it} K^{\beta}_{it} e^{U_i} \qquad (1)$$

$$i = 1,2,3,\ldots.., N.$$

$$t = 1,2,3,\ldots.,T$$

Where:

Q_{it} = Total production i at time t (the monetary value of all goods produced in a year) : Outputs.

L_{it} = Labor i at time t: input.

K_{it} = Capital i at time t: input.

A_i = Intercept Term (total factor productivity), and the index "i" indicate that the intercepts are differ from time to time .

U= Error term with $E(U)=0 \quad Var(U)=\sigma_U^2$ and $E(U_iU_j)=0 \quad for \;\; i \neq j$

α and β are the output elasticity of labor and capital, respectively. These values are constants determined by available technology.

2. Log Transformation[5,6]:

Taking Logs , the model of equation (1) becomes:

$$Log\,Q_{it} = LogA_i + \alpha \log L_{it} + \beta \log K_{it} + U_{it} \qquad (2)$$

where *logA* is the intercept. Now suppose that we want to estimate this model under the restriction of homogeneity of degree 1, i.e., if both *K* and *L* increase with say 10% then so wills *Q*. This condition is equivalent to $\alpha + \beta = 1$.

Thus, in equation (2) replace α with $1-\beta$ we get:

$$Log\,Q_{it} = LogA_i + (1-\beta)\log L_{it} + \beta \log K_{it} + U_{it} \qquad (3)$$

This model can be reformulated and estimated as an unrestricted linear regression model, as follows:

$$Y_{it} = Log\,Q_{it} - \log L_{it} = LogA_i + \beta(\log K_{it} - \log L_{it}) + U_{it} \qquad (4)$$

Then,

$$Y_{it} = \beta_{oi} + \beta_1 X_{it} + U_{it} \qquad (5)$$

Where,

$Y_{it} = LogQ_{it} - LogL_{it}$, $\beta_{0i} = LogA_i$, $X_{it} = LogK_{it} - LogL_{it}$

3. Alternative transformation for Cobb Douglas function:

In this section we suggest an alternative method to transform the standard Cobb Douglas production function into linear function by introducing dummy variables. The technique of dummy variables used by Walnace and Ashiq[8] to combining cross sector with time series data, and this method is also used by Hussein Y. and Abdallah Kh. to find out linear function for the Cobb Douglas production function [for example see 4].

Now we can rewrite model of equation (5) as:

$$Y_{it} = \sum_{i=1}^{N} \beta_{oi} D_{it} + \beta_1 X_{it} + U_{it} \qquad (6)$$

Where D_{it} is the dummy variable and defined as:

$$D_{it} = \begin{cases} 0 & if \quad i \neq t \\ 1 & if \quad i = t \end{cases}$$

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Equation (6) can be written as:

$$Y_{it} = \beta_{o1} D_{1t} + \sum_{i=2}^{N} \beta_{oi} D_{it} + \beta_1 X_{it} + U_{it}$$

$$= \beta_{o1} D_{1t} + \sum_{i=2}^{N} (\beta_{o1} + \delta_i) D_{it} + \beta_1 X_{it} + U_{it} \text{ where } \beta_{oi} = \beta_{o1} + \delta_i$$

$$= \beta_{o1} \sum_{i=1}^{N} D_{it} + \sum_{i=2}^{N} \delta_i D_{it} + \beta_1 X_{it} + U_{it}$$

$$= \beta_{o1} + \sum_{i=2}^{N} \delta_i D_{it} + \beta_1 X_{it} + U_{it} \qquad (7)$$

Note: The last step obtained by taking the constraint $\sum_{i=1}^{N} D_{it} = 1$

To simplified the processes of estimating a parameters of the model its better to transform it into matrix notation .Now we can rewrite equation (6) namely $Y_{it} = \sum_{i=1}^{N} \beta_{oi} D_{it} + \beta_1 X_{it} + U_{it}$ In matrix notation as:

$$Y = \beta_o j + \underline{X}_i \beta_1 + U \qquad (8)$$

Where:

$$\underset{(T\times N)}{Y} = [Y_{i1}, Y_{i2}, \ldots\ldots\ldots Y_{iT}]'$$

$$\underset{(T\times 1)}{j} = [1,1,\ldots\ldots\ldots\ldots 1]'$$

$$\underset{(T\times N)}{X} = [X_{i1} \quad X_{i2} \ldots\ldots\ldots\ldots \quad X_{iT}]'$$

$$\underset{(T\times N)}{U} = [U_{i1} \quad U_{i2} \ldots\ldots\ldots\ldots\ldots\ldots U_{iT}]'$$

Or in more compact form as:

$$Y = [I_N \otimes J_T \qquad X]\begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix} + U$$

$$or \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} J_T & & 0 & X_1 \\ & J_T & & X_2 \\ & & \ddots & \vdots \\ 0 & & J_T & X_N \end{bmatrix} \begin{bmatrix} \beta_{01} \\ \beta_{02} \\ \vdots \\ \beta_{0N} \\ \beta_1 \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} \qquad (9)$$

Now equation (9) takes the form of the general linear mode as:

$$Y = X\beta + U \qquad (10).$$

Where,

$$Y = [Y_1 Y_2 \ldots\ldots. Y_N]' \qquad X = \begin{bmatrix} I_N \otimes J_T & X \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_0 \\ - \\ \beta_1 \end{bmatrix} \qquad U = [U_1 U_2 \ldots\ldots\ldots.. U_N]$$

Then, Appling ordinary least square method to estimates $\hat{\beta}$ we get:

$$\hat{\beta} = (X'X)^{-1} X'Y$$

$$ie \qquad \hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \left[\begin{bmatrix} (I_N \otimes J_T)' \\ X' \end{bmatrix} [I_N \otimes J_T \qquad X] \right]^{-1} \begin{bmatrix} (I_N \otimes J_T)' \\ X' \end{bmatrix} Y$$

$$= \begin{bmatrix} T_{I_N} & (I_N \otimes J_T)' X \\ X'(I_N \otimes J_T) & X'X \end{bmatrix}^{-1} \begin{bmatrix} (I_N \otimes J_T)' Y \\ X'Y \end{bmatrix} \qquad (11)$$

But if N is large the inversion of X in equation (10) will be unreliable, so an alterntive expression is suggested for estimating "$\hat{\beta}_0$" and "$\hat{\beta}_1$": [7]

$$\hat{\beta} = [(X')(I_N \otimes D_T)X \quad)]^{-1} X'(I_N \otimes D_T)Y \qquad (12)$$

Where: $D_T = I_T - \frac{J_T J_T'}{T}$ and $\hat{\beta}_{0i} = \overline{Y}_i - \overline{X}_{i.}\hat{\beta}_1$

where : $\overline{Y}_i = \frac{1}{T}\sum_{t=1}^{T} Y_{it}$, $\overline{X}_i = \frac{1}{T}\sum_{t=1}^{T} X_{it}$

(This alternative expressions are obtained from equation (11) by using the partition inverse)

And to simplify the computation procedure in practice, it can be shown that D_T is idempotent and hence $(I_N \otimes D_T)$ is also idempotent. Therefore, equation (12) can be written as:

$$\hat{\beta} = \left(X'(I_N \otimes D_T)'(I_N \otimes D_T)X\right)^{-1} X'(I_N \otimes D_T)'(I_N \otimes D_T)Y$$

$$= (Z'Z)^{-1} Z'W \qquad (13)$$

Where:

$$Z = (I_N \otimes D_T)X \qquad (14)$$

$$W = (I_N \otimes D_T)Y \qquad (15)$$

Then, transform the observation on the independent variable and the dependent variable as follows:

equation (14) will be:

$$Z = (I_N \otimes D_T)X = \begin{bmatrix} D_T & & & O \\ & D_T & & \\ & & \ddots & \\ O & & & D_T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} D_T X_1 \\ D_T X_2 \\ \vdots \\ D_T X_N \end{bmatrix}$$

Examining the i^{th} vector:

$$D_T X_i = \left(I_T - \frac{J_T J_T'}{T}\right) X_i = X_i - \overline{X}i.J_T$$

Since $\frac{J_T X_i}{T} = \overline{X}_{i.}$

$$D_T X_{it} = X_{it} - \overline{X}_{i.} = X^*_{it} \quad (16)$$

And similarly, equation (15) will be:

$$W = (I_N \otimes D_T)Y \Rightarrow$$
$$D_T Y_{it} = Y_{it} - \overline{Y}_{i.} = Y_{it}^* \quad (17)$$

Now the transformed model is:

$$Y^*_{it} = X^*_{it}\beta + U^*_{it} \qquad \text{where} \qquad U^*_{it} = U_{it} - \overline{U}_{i.} \quad (18)$$

Then the ordinary least square estimate for β is:

$$\hat{\beta} = \left(X^* X^*\right)^{-1} X^* Y^* \quad (19)$$

4. Numeric example

We now illustrate our theoretical results which represents the new linear transformation and compare with traditional one (log transform) with observations consist of 19 points through 1982-2000, due to Ministry of Tourism in Egypt (2000). Where Q (Output) is the number of tourist nights, L (Labor) is the number of workers in tourism sector and K (Capital) is the number of tourist rooms in all Hotels in Egypt [4].

Table 1: Data for Tourism Sector

Year	**Tourist Nights**	**Tourist Rooms**	**Labor in Tourism**
1982	9301890	18864	104000
1983	8856746	22312	108000
1984	8572200	24710	112000
1985	9007053	27311	116000
1986	7847866	31476	104000
1987	15861403	33032	122000
1988	17863995	34603	128000
1989	20582680	39809	136000
1990	19942388	51208	143000
1991	16104499	53727	147000
1992	21835705	55610	151000
1993	15089834	58755	130000
1994	15432753	61068	133000
1995	20451364	64958	136000
1996	23764641	70471	140000
1997	26578830	75679	145000
1998	20150520	82925	145000
1999	31002088	93822	147000
2000	32787880	114000	150000

Source: (Khaled ,Abd.,M. Shawadfy:2005)

Our computations here were performed by using SPSS17, and our analysis concentrated into how to compute the two method of transformation instead of the economical interpretations.

When apply data in table1 to the traditional log transformation model (5) namely $Y_{it} = \beta_{oi} + \beta_1 X_{it} + U_{it}$ we obtain the results appear in the following table 2 below:

Table 2: Results of the Log transformation Model

Model	$\hat{\beta}$	**P.value**		
intercept	2.387	0.00000	R^2	0.634
Slope	0.652	0.00004	MSE	0.008
F	29.391	0.00005		

Source: Founded by Author

From table 2 above we note that, the estimated model $\hat{Y}_{it} = \hat{\beta}_{oi} + \hat{\beta}_1 X_{it}$ is :

$$\hat{Y}_{it} = 2.387 + 0.652 X_{it}$$

And P.values of the regression coefficients indicates that β_0 and β_1 are statistically significantly different from zero , whereas P.value of F means that the model is significant and there is a linear relationship between transformed variables Y (dependent) and X (independent) .

When apply data in table1 to the alternative transformation model (18) namely $Y^*_{it} = X^*_{it}\beta + U^*_{it}$ we obtain the results appear in the following table 3 below :

Table 3: Results of the Alternative transformation Model

Model	$\hat{\beta}$	**P.value**		
intercept	-3.8E-16	1.0000	R^2	0.634
Slope	0.652	0.00004	MSE	0.008
F	29.391	0.00005		

Source: Founded by Author

From table 3 we note that, the results of the alternative transformation model are equivalent to that of the traditional log transformation model except the intercept of the model $\hat{\beta}_o$ which is almost equal zero , and parameter β_0 is not significant . So the estimated model $\hat{Y}_{it} = \hat{\beta}_{oi} + \hat{\beta}_1 X_{it}$ is:

$$\hat{Y}_{it} = 0.652 X_{it}$$

5. results:

1. The alternative transformation method equivalent to the traditional log transformation.
2. The constant term of the alternative transformation model is not significant, which is a logic result since the constant represents the

amount of the independent variable "Y" when the explanatory variable "X" is zero and if X = 0 "Y" will be ∞ - because of the opposite relation between them - which is not logical in reality. so the constant term will be omitted from the model and the vector "b" in equation (17) will be: $\begin{pmatrix} \underline{0} \\ \\ \underline{\beta_i} \end{pmatrix}$

6. Conclusion:

From the results obtained in this study, we can conclude that the new transformation model obtained is an alternative for the traditional log model.

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