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CR-Submanifolds of Kaehlerian Manifolds

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مستخلص البحث:

في هذه الورقة درسنا كثيرات الطيات الجزئية لـ (كوشي وريمان) في كثيرات طيات (كهلر)، استنتجنا فيها معادلات (جاوس وكوداسي وريسي)، ثم استخدمنا مفردات هذه المعادلات في تجميع بعض النظريات المميزة التصنيفية.

1. Abstract

In this paper we considered the concept of CR-submanifolds of Kaehlerian manifolds. We introduced the compatibility equations of Gauss, Codazzi and Ricci. Then we utilized the above equations to deduce some characterization theorems for CR-submanifolds

2. Introduction

The notion of CR-submanifolds was introduced by A. Bejanco [4] and then several publications have paved the way to acquire knowledge about the characterization of CR-submanifolds embedded in different manifolds [1, 2, 3] and [9]. In this paper we considered CR-submanifolds of Kaehlerian manifolds. First we treated Kaehlerian manifold and introduced the complex structure. It is known that a manifold need not be totally real or complex. So the notion of CR-submanifold came into play. We defined this notion of CR-

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submanifolds, then we treated the compatibility equations of Gauss, Codazzi and Ricci that are adapted to our study. At the end of the paper we considered the characterization problem of CR-submanifolds.

3. Kaehlerian manifolds

3.1. Basic Concepts

In this section we give the fundamental concepts concerning the study.

Let $\tilde{M}$ be a Riemann manifold and M be a submanifold of $\tilde{M}$. The Riemannian metric g on $\tilde{M}$ induces a Riemannian metric on M. Let TM and $TM^{\perp}$ denote tangent and normal bundles, respectively, and $\bar{\nabla}$, ∇ be the Levi-Civita connections on $\tilde{M}$ and M, respectively, then for $X, Y \in \Gamma(TM)$ we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad (3.1)$$

where $\Gamma(TM)$ is the module of differentiable sections defined on the bundle TM and h is the second fundamental form of M. The equation above is called as the Gauss formula. V being an element of $\Gamma(TM^{\perp})$ the Wiengarten formula is given by

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \qquad (3.2)$$

where A_V is the fundamental tensor of Weingarten with respect to the normal section V, and $\nabla^{\perp}$ is the normal connection on M. It is well known that

$$g\left(h(X, Y), V\right) = g(A_V X, Y) \; (3.3)$$

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for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$.

A q-dimensional distribution on an n-dimensional manifold M is a mapping D defined on M which assigns to each point x of M a q-dimensional linear subspace D_X of $T_X(m)$.

D is said to be differentiable if there exist q differentiable vector fields on a neighborhood of x, for each point (y) in this neighborhood of x, which form a basis of D_Y. The set of these q vector fields is called a local basis of D.

An almost complex structure on a differentiable manifold M is a tensor field J of type (1.1) which is at every point x of M, an endomorphism of $T_X(m)$ such that $J^2 = -1$, where I denotes the identity transformation of $T_X(m)$.

A manifold M which an almost complex structure J is called an almost complex manifold.

The torsion of the an almost complex structure J is a tensor field N of type (1.2) called the Nijenhuis torsion given by :

$$N(X, Y) = \left([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\right) \quad (3.4)$$

for any vector fields X and Y.

An almost complex structure J is called a complex structure if it is torsion N vanishes identically and M is called a complex manifold.

A Hermition metric on almost complex manifold M is a Riemannian metric g such that

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$$g(JX, JY) = g(X,Y) \qquad (3.5)$$

for any vector field X and Y.

An almost complex manifold (resp. A complex manifold) with Hermition metric is called an almost Hermition manifold (resp. Hermition).

We notice that every almost complex manifold M with a Riemannian metric g admits a Hermition metric. Indeed, for any almost complex structure J on M putting

$$h(X,Y) = g(X,Y) + g(JX, JY) \quad (3.6)$$

for any vector fields X and Y we obtain a Hermition metric h.

A Hermition manifold M is called a Kaehlerian manifold if the almost complex structure J of N is parallel, that is $\nabla J = 0$.

4. Gauss, Ricci and Codazzi equations

Let $\tilde{M}$ be a complex m-dimensional (2m-dimensional) Kaehlerian manifold with almost complex structure J and with Kaehlerian metric g. Let M be a real n-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. We denote by the same g the Riemannian metric tensor field induced on M from that $\tilde{M}$. The operator of covariant differentiation in $\tilde{M}$ (resp. M) will denote by $\tilde{\nabla}(resp.\nabla)$.

For any vector field X tangent to M, we put

$$JX = PX + FX \qquad (4.1)$$

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where PX is the tangent part of JX and FX is the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle I-from on the tangent bundle $T(M)$.

For any vector field V normal to M we put

$$JV = tV + fV \qquad (4.2)$$

where tV is the tangential part of JV and fV the normal part of JV. For any vector field Y tangent to M, we have from (4.1), $g(JX,Y) = g(PX,Y)$, which shows that $g(PX,Y)$ is a skew-symmetric. Similarly, for any vector U normal to M, we have, from (4.2), $g(JV,U) = g(fV,U)$, which shows that $g(fV,U)$ is a skew-symmetric.

From (4.1) and (4.2) we also have

$$g(FX,V) + g(X,tV) = 0 \qquad (4.3)$$

which gives the relation between F and t.

Now, applying J to (4.1) and using (4.1) and (4.2) we find

$$P^2 = -I - tF, \qquad FP + fF = 0 \quad (4.4)$$

Applying J to (4.2) and using (4.1) and (4.2) we find

$$Pt + tf = 0, \qquad f^2 = -I - Ft \qquad (4.5)$$

We define the covariant derivative $\nabla_X P$ of P by

$$(\nabla_X P)Y = \nabla_X (PY) - P\nabla_X Y$$

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And the covariant derivative $\nabla_X F$ of F by $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$. Similarly, we define the covariant derivative $\nabla_X t$ of t and $\nabla_X f$ of f by $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$ respectively. Then, from the Gauss and Weingarten formulas we have

$$tB(X,Y) + fB(X,Y) = (\nabla_X P)Y - A_{FX}X + B(X,PY) + (\nabla_X F)Y$$

Comparing the tangential and normal parts of both sides of this equation, we find

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y), \qquad (4.6)$$

$$(\nabla_X F)Y = -B(X,PY) + fB(X,Y), \qquad (4.7)$$

Similarly, we have

$$-PA_V - FA_V X = (\nabla_X t) - A_{fX}X + B(X,tV) + (\nabla_X f)V$$

, from which

$$(\nabla_X t)V = A_{fX}X - PA_V X, \qquad (4.8)$$

$$(\nabla_X f)V = -FA_V X - tB(X,tV) \qquad (4.9)$$

Let M be an n-dimensional submanifold of a complex space from $M^{-m}(c)$. Then the curvature tensor R of M is given by

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$$R(X,Y)Z=\frac{1}{4}c\,[g(Y,Z)X-g(X,Z)Y$$
$$+g(JY,Z)JX-g(JX,Z)JY$$
$$+2g(X,JY)JZ]+A_{B(Y,Z)}X-A_{B(X,Z)}Y$$
$$+(\nabla_Y B)(X,Z)-(\nabla_X B)(Y,Z)$$

For any vector field X,Y and Z tangent to M. Comparing the tangential and normal part of the both sides of this equation, we have, following equations of Gauss and Codazzi respectively

$$R(X,Y)Z=\frac{1}{4}c\left[g(Y,Z)X-g(X,Z)Y+g(PY,Z)PX\right.$$

$$\left.-g(PX,Z)PY+2g(X,PY)PZ\right]$$
(4.10)
$$+A_{B(Y,Z)}X-A_{B(X,Z)}Y\;,$$

$$(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)$$
$$=\frac{1}{4}c\left[g(PY,Z)FX-g(PX,Z)FY+2g(X,PY)FZ\right]$$
(4.11)

Similarly, we have the equation of Ricci:

$$g(R^{\wedge}(X,Y)U,V)+([A_V,A_U]X,Y)=$$

$$\tfrac{1}{4}c[g(FY,U)g(FX,V)-g(FX,U)g(FY,V)$$

(4.12)

$$+\ 2g(X, PY)\,g(fU, V)$$

5. CR-Submanifold of Kaehler manifold

Let $\tilde{M}$ be a Kaehlerian manifold with almost complex structure J, A submanifold of $\tilde{M}$ is called CR-submanifold of $\tilde{M}$ if there exists a differentiable distribution $D : x \longrightarrow D_X \subset T_X(M)$ on M satisfying the following conditions:

i) D in invariant, i.e., $JD_X = D_X$ for each $x \in M$ and

ii) the complementary orthogonal distribution

$D^{\perp} : x \longrightarrow D_X^{\perp} \subset T_X(M)$ is anti-invariant, i.e.,

$J\,D_X^{\perp} \subset T_X(M)^{\perp}$ for each $x \in M$

In the sequel, we put $\dim \tilde{M} = 2m$, $\dim m = n$, $\dim D = h$, $D^{\perp} = q$ and codim $M = 2m - n = P$. If $q = 0$, then a CR-submanifold is called an invariant submanifold of $\tilde{M}$, and if $h = 0$, then M is called an anti-variant submanifold of $\tilde{M}$.

If $P = q$, then a CR-submanifold M is called a generic submanifold of $\tilde{M}$. If $h > 0$ and $q > 0$ then a CR submanifold M is said to be non-trivial (proper).

If M is an invariant submanifold of a Kaehlerian $\tilde{M}$, F in (4.1) vanishes identically. Moreover, we see that t in (4.2) vanished identically. Thus we have $J\,X = FX$ and $J\,V = f\,V$. From (4.6)

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we see that any invariant submanifold of a Kaehlerian manifold is also Kaehlerian manifold with respect to induced structure. From (4.7) and (4.8) we have

***Lemma* 5.1.** Let M be an invariant submanifold of Kaehlerian submanifold $\tilde{M}$. Then

$$B(X,JY)=B(JX,Y)=JB(X,Y) \qquad (5.1)$$

$$JA_V+A_VJX=0 \qquad (5.2)$$

$$A_{JV}X=JA_VX \qquad (5.3)$$

***Theorem* 5.1**: In order for a submanifold M of a Kaehlerian manifold $\tilde{M}$ to be a CR submanifold, it is necessary and sufficient that $FP=0$.

***Theorem* 5.2**: Let M be a CR-submanifold of a Kaehlerian manifold $\tilde{M}$ then P is an f-structure in M and f is an f-structure the normal bundle of M.

***Lemma* 5.2**: Let M be a CR-submanifold of a Kaehlerian manifold $\tilde{M}$. Then we have

$$A_{FX}Y=A_{FY}X$$

***Theorem* 5.3.** Let M be a CR submanifold of a Kaehlerian manifold $\tilde{M}$. Then the distribution D is integrable if and only if the second fundamental form satisfies

$$h(X,JY)=h(JX,Y) \quad \text{for } X,Y\in\Gamma(D)$$

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***Theorem* 5.4**. Let M be a CR submanifold of a Kaehlerian manifold $\tilde{M}$. Then the distribution $D^{\perp}$ is completely integrable and its maximal integral submanifold $M^{\perp}$ is an anti-invariant submanifold of $\tilde{M}$.

***Theorem* 5.5.** Let M be a CR-submanifold of a Kaehlerian manifold $\tilde{M}$. Then the f-structure P is partially integrable if and only if

$$B(PX,Y)=B(X,PY) \qquad (5.4)$$

***Lemma* 5.3.** Let M be a mixed foliate CR-submanifold of Kaehlerian manifold $\tilde{M}$. Then we have

$$A_V P + PA_V = 0 \qquad (5.5)$$

for any vector field V normal to M .

5.1. CR-product in Kaehler manifolds

A CR-submanifold of a Kaehler manifold $\tilde{M}$ is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold N^T and a totally real submanifold $N^{\perp}$ of $\tilde{M}$.

***Theorem* 5.1.1. [Chen, 1981].**

A CR-submanifold of a Kaehler manifold is a CR-product if and only if P is parallel.

***Theorem* 5.1.2. [Chen, 1981].** A CR-submanifold of a Kaehler manifold is a CR-product if and only if

$$A_{JD^{\perp}} D = 0 .$$

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***Lemma* 5.1.1.** Let M be a CR-product of a Kaehler manifold $\tilde{M}$. Then for any unit vectors $X \in D$ and $Z \in D^{\perp}$ we have

$$\tilde{H}_B(X,Z) = 2\|B(X,Z)\|^2$$

where $\tilde{H}_B(X,Z) = \tilde{g}(Z, \tilde{R}_{X,JX} JZ)$ is the holomorphic bisectional curvature of the plane $X > Z$.

***Theorem* 5.1.3. (Chen, 1981).** Let $\tilde{M}$ be a Kaehler manifold with negative holomorphic bisectional curvature. Then every CR-product in $\tilde{M}$ is either a holomorphic submanifold or a totally real submanifold. In particular, there exists no proper CR-product in any complex hyperbolic space $\tilde{M}(c), (c < 0)$.

A warped product in CR-submanifold of Kaehler manifold defined as $M = N^{\perp} \times_f N^T$ (i.e. If $(B, g_B), (F, g_F)$ Riemannian manifolds, $f > 0$ smooth function on B, $M = B \times_f F, g = g_B + f^2 g_F$.

***Theorem* 5.1.4. (Chen, 2001).** If $M = N^{\perp} \times_f N^T$ is a warped product CR-submanifold of a Kaehler manifold $\tilde{M}$ such that $N^{\perp}$ is a totally real submanifold and N^T is a holomorphic submanifold of $\tilde{M}$, then M is a CR-product.

***Remark* (Chen, 2001).** There do not exist warped product CR-submanifolds in the for $N^{\perp} \times_f N^T$ other than CR-products.

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By contrast, there exist many warped product CR-submanifolds $N^T \times_f N^\perp$ which are not CR-products.

***Theorem* 5.1.5. (Chen, 2001).** A proper CR-submanifold M of a Kaehler manifold $\tilde{M}$ is locally a CR-warped product if and only if

$$A_{JZ}Z = ((JX)\mu)Z, \qquad X \in D, \quad Z \in D^\perp$$

for some function μ on M satisfying $W\mu = 0$, for all $W \in D^\perp$.

***Theorem* 5.1.6. (Chen, 2001).** Let $M = N^T \times_f N^\perp$ be a CR-warped product in a Kaehler manifold $\tilde{M}$. Then

1. $\|B\|^2 \geq 2q\|\nabla(\log f)\|^2$, where $\nabla(\log f)$ is the gradient of $\log f$,

2. If the equality sign holds identically, then N^T is a totally geodesic and $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$. Moreover, M is a minimal submanifold in $\tilde{M}$.

3. When M is generic and $q > 1$, the equality sign holds if and only if $N^\perp$ is a totally umbilical submanifold of $\tilde{M}$.

4. When M is generic and $q = 1$, then the equality sign holds if and only if the characteristic vector of M is a principal vector field with zero as its principal curvature. (In this case M is a real hypersurface in $\tilde{M}$).

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A Twisted product in CR-submanifold of Kaehler manifold is defined as $M = N^{\perp} \times_f N^T$ (i.e. If $(B, g_B), (F, g_F)$ are Riemannian manifolds, $f > 0$ smooth function on $B \times f$, then $M = B \times_f F, g = g_B + f^2 g_F$.

Theorem 5.1.7. (Chen, 2000). If $M = N^{\perp} \times_f N^T$ is a twisted product CR-submanifold of a Kaehler manifold $\tilde{M}$ such that $N^{\perp}$ is a totally real submanifold and N^T is a holomorphic submanifold of $\tilde{M}$, then M is a CR-product.

Theorem 5.1.8. (Chen, 2000). Let $M = N^T \times_f N^{\perp}$ be a CR-warped product in a Kaehler manifold $\tilde{M}$. Then

1. $\|B\|^2 \geq 2q \|\nabla^T (\log f)\|^2$, where $\nabla^T (\log f)$ is the N^T–component of the gradient of log f,

2. If the equality sign holds identically, then N^T is a totally geodesic and $N^{\perp}$ is a totally umbilical submanifold of $\tilde{M}$.

3. When M is generic and $q > 1$, the equality sign holds if and only if N^T is a totally geodesic and $N^{\perp}$ is a totally umbilical submanifold of $\tilde{M}$.

6. Conclusion and future outlook

Historically CR-submanifolds are related to variations, possibly described by partial differential equation. Thus the classifications of

CR-submanifolds are linked to the classification of solution of this equations. The point which is not consider in this paper is whether some classifications of CR-submanifolds come from variational principles. This will be our future outlook.

References

1. Bashir, M.A., Chen's Problem on Mixed Foliate CR-submanifolds, Bull. Austral. Math. Soc., Vol. 40 (1989), 157-160.

2. Bashir, M.A., On CR-submanifolds of six-dimensional sphere, Internat. J. Math. & Math. Sci., Vol. 18 No. 1 (1995), 201-203.

3. Bashir, M.A., On the three-dimensional CR-submanifolds of the six-dimensional sphere, Internat, J. Math. & Math., Sci., 14 (1991), 675-678.

4. Bejancu, A., CR-submanifolds of a Kaehler manifold, Proc. Amer. Math. Soc. 69 (1978), 135-142.

5. Chen, B. Y., Geometry of submanifolds, Marcel Dckker Inc., New York, 1973.

6. Chern, S. S., Chen, W.H. and Lam, K.S., Lectures on Differential Geometry, World Scientific Publishing Co., Pte., Ltd., 1999.

7. MUNTEANU, M.I., On the geometry of CR-submanifolds of product type, Workshop, University of Luxembourg, March 24 – 26, 2009

8. Sahin, B., Gunes, R. & Sivridag, A. I., On CR-Submanifold having holomorphic vector fields on them, Common, Fac. Sci. Univ. Ank. Series A1, V. 47, pp. 27-34, 1998.

9. Sekigawa, K., Some Cr-submanifolds in a 6-dimensional sphere, Tensor, N.S., 41(1984), 13-20.

10. Yano, K., Kon, M., CR Submanifolds of Kaehlerian and Sasakian Manifolds, Boton Inc., 1983.